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Range kernel orthogonality of derivations

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Abstract

Let H be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of operators on H into itself. The generalised derivation $\delta_{A,B} : B(H) \to B(H)$ is defined by $\delta_{A,B}(X) = AX - XB$; let $\triangle_{A,B} : B(H) \to B(H)$ be defined by $\triangle_{A,B}(X) = AXB - X$, and let $d_{A,B}$ denote $\delta_{A,B}$ or $\triangle_{A,B}$. Let $S \in \mathscr{C}_p$ (the Schatten p-class, $1 < p < \infty$). Given that the pair (A, B) of operators satisfies the property $\ker d_{A,B}|\mathscr{C}_p \subseteq \ker d_{A^*,B^*}|\mathscr{C}_p$, we prove a necessary and sufficient condition for $\|d_{A^*,B^*}(X) + S\|_p \geqslant \|S\|_p$ to hold for all $X \in \mathscr{C}_p$.

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Let $B(H)$ denote the algebra of operators (= bounded linear transformations) on a separable infinite dimensional complex Hilbert space H into itself. Given $A, B \in B(H)$, we define the generalised derivation $\delta_{A,B} : B(H) \to B(H)$ by $\delta_{A,B}(X) = AX - BX$, and (the related) derivation $\triangle_{A,B} : B(H) \to B(H)$ by $\triangle_{A,B}(X) = AXB - X$. Let $\delta_{A,A} = \delta_A$, $\triangle_{A,A} = \triangle_A$, and let $d_{A,B}$ denote either $\delta_{A,B}$ or $\triangle_{A,B}$. Given subspaces M and N of a Banach space V with norm $\|\cdot\|$, M is said to be orthogonal to N if $\|m+n\| \geqslant \|n\|$ for all $m \in M$ and $n \in N$. Let A be a normal operator. Anderson [1] has shown that if S is in the commutatant of A (i.e., $[A, S] = AS - SA = 0$), then $\|\delta_A(X) + S\| \geqslant \|S\|$ for all $X \in B(H)$, i.e., the range $\operatorname{ran} \delta_A$ of the derivation δ_A is orthogonal to the kernel $\ker \delta_A$ of δ_A. This range–kernel orthogonality result has a $\triangle_A$ analogue [9,10].

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Let $X \in B(H)$ be compact, and let $s_1(X) \geqslant s_2(X) \geqslant \cdots \geqslant 0$ denote the singular values of X arranged in their decreasing order. The operator X is said to belong to the Schatten p-class $\mathscr{C}_p$ if

$$\|X\|_p = \begin{cases} \left(\sum_{j=1}^{\infty} s_j(X)^p\right)^{1/p} = (\mathrm{tr}(X)^p)^{1/p} < \infty, & 1 \leqslant p < \infty, \\ s_1(X), & p = \infty, \end{cases}$$

where 'tr' denotes the trace functional. The range–kernel orthogonality of $d_{A,B}|\mathscr{C}_p$ ($= d_{A,B}$ restricted to $\mathscr{C}_p$), and more generally for the class of unitarily invariant norms, has been considered in a number of papers [1,3,5,7,8,9,12–15] (see also [2,4]). A typical result here assumes the normality of A and B (or restrictions of A and B to some suitably defined reducing subspaces), and then proceeds to show that if $d_{A,B}(S) = 0$ for some $S \in \mathscr{C}_p$ then $\|d_{A,B}(X) + S\|_p \geqslant \|S\|_p$ for all $X \in \mathscr{C}_p$. A characterization of $S \in \mathscr{C}_p$, $1 < p < \infty$, which are orthogonal to ran $\delta_A|\mathscr{C}_p$ for a general operator A has been carried out by Kittaneh [14]: if S has the polar decomposition $S = U|S|$, then $\|\delta_A(X) + S\|_p \geqslant \|S\|_p$ for all $X \in \mathscr{C}_p$ $(1 < p < \infty)$ if and only if $\delta_A(|S|^{p-1}U^*) = 0$. In the case in which A is normal, $\delta_A(|S|^{p-1}U^*) = 0 \Leftrightarrow \delta_A(S) = 0$; hence $\|\delta_A(X) + S\|_p \geqslant \|S\|_p$ for normal A and all $X \in \mathscr{C}_p$ $(1 < p < \infty)$ if and only if $\delta_A(S) = 0$. That the "if" part of this result extends to $\delta_{A,B}$ for normal A and B has been proved by Bouali and Cherki [5] (see also [15]).

Let I denote a two sided ideal of $B(H)$. We say that the pair of operators (A, B) is a PF(d, I) pair if ker $d_{A,B}|I \subseteq$ ker $d_{A^*,B^*}|I$. (Here PF is short for Putnam–Fuglede property : (A, B) is a PF(d, I) pair if, for a given $S \in I$, $d_{A,B}(S) = 0 \Longrightarrow d_{A^*,B^*}(S) = 0$ [6].) It is known that the hypothesis $d_{A,B}(S) = 0$, $S \in I$, for some PF(d, I) pair (A, B) is sufficient for the inequality $\|d_{A,B}(X) + S\|_I \geqslant \|S\|_I$ to hold for all $X \in I$ [3,5,9,13,15]. In their consideration of the derivation $\delta_{A,B}|\ \mathscr{C}_p$, Bouali and Cherki [5] ask if the condition $\delta_{A,B}(S) = 0$ is also necessary. We consider this problem here, and we prove:

Theorem.

(i) *If* ker $d_{A,B} \subseteq$ ker d_{A^*,B^*}, *then*

$$\min\left\{\|d_{A,B}(X) + S\|,\ \|d_{A^*,B^*}(X) + S\|\right\} \geqslant \|S\| \tag{1}$$

for all $S \in$ ker $d_{A,B}$ *and* $X \in B(H)$.

(ii) *Suppose that* ker $d_{A,B} \mid \mathscr{C}_p \subseteq$ ker $d_{A^*,B^*} \mid \mathscr{C}_p$; $1 < p < \infty$. *Then, given* $S \in \mathscr{C}_p$,

$$\|d_{A^*,B^*}(X) + S\|_p \geqslant \|S\|_p \tag{2}$$

for all $X \in \mathscr{C}_p$ *if and only if* $S \in$ ker $d_{A,B}$.

(iii) *Let* $A, B \in B(H)$, *and let* $1 < p < \infty$. *Given* $S \in \mathscr{C}_p$,

$$\min\left\{\|d_{A,B}(X) + S\|_p,\ \|d_{A^*,B^*}(X) + S\|_p\right\} \geqslant \|S\|_p \tag{3}$$

for all $X \in \mathscr{C}_p$ *if and only if* $S \in$ ker $d_{A,B} \cap$ ker d_{A^*,B^*}.

Inequality (1), even though rarely stated in this form, is well-known for the case in which A, B are normal operators (see [9,13]); part (ii) of the theorem says that if the pair (A, B) has the PF$(d, \mathscr{C}_p)$ property, then (2) holds for a given $S \in \mathscr{C}_p$ and all $X \in \mathscr{C}_p$ if and only if $d_{A,B}(S) = 0$ (thereby showing that the answer to the Bouali–Cherki problem is in the affirmative in the sense that (1) holds if $S \in \ker d_{A,B}$ and $\ker d_{A,B}|\mathscr{C}_p \subseteq \ker d_{A^*,B^*}|\mathscr{C}_p$, and that if $\ker d_{A,B}|\mathscr{C}_p \subseteq \ker d_{A^*,B^*}|\mathscr{C}_p$ and (1) holds, then $S \in \ker d_{A,B}$); part (iii) of the theorem is a sort of converse to the PF-property hypothesis in (ii) – it says that if (3) holds for a set E of $S \in \mathscr{C}_p$, then the restriction of $d_{A,B}$ to E has the Putnam–Fuglede property.

The proof of the theorem proceeds through a number of steps, stated below as lemmas.

Recall that the norm $\|\cdot\|$ of the B-space V is said to be Gateaux differentiable at a non-zero element $x \in V$ if

$$\lim_{\substack{t\in\mathbb{R}\\ t\to 0}} \frac{\|x+ty\| - \|x\|}{t} = \Re D_x(y),$$

for all $y \in V$. Here $\mathbb{R}$ denotes the set of reals, $\Re$ denotes the real part and D_x is the unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gateaux differentiability of the norm at x implies that x is a smooth point of the sphere with radius $\|x\|$.

Lemma 1 [11,14]. *Let $x, y \in V$. If x is a smooth point of V, then $D_x(y) = 0 \Leftrightarrow \|x+ty\| \geqslant \|x\|$ for all complex numbers t.*

Let $S \in \mathscr{C}_p$ have the polar decomposition $S = U|S|$. Define the operators $\hat{S}$ and $\tilde{S}$ by

$$\hat{S} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{S} = \begin{bmatrix} 0 & 0 \\ |S|^{p-1}U^* & 0 \end{bmatrix}$$

on $\hat{H} = H \oplus H$. Then

$$\|\tilde{S}\|_{p/(p-1)} = [\mathrm{tr}(U|S|^{p-1}U^*)^{p/(p-1)}]^{(p-1)/p} = [\mathrm{tr}(U|S|^{p}U^*)]^{(p-1)/p} = \|S\|_p^{p-1},$$

i.e., $\tilde{S} \in \mathscr{C}_q$, where q is the index conjugate to p.

Lemma 2 (cf. [14, Theorem 2]). *Let $A, B \in B(H)$. Given $S \in \mathscr{C}_p$, $1 < p < \infty$, inequality* (2) *holds for all $X \in \mathscr{C}_p$ if and only if $\tilde{S} \in \ker d_{T,T^*}$, where $T = A \oplus B^*$.*

Proof. Define the operator Y, on $H \oplus H$, by

$$Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix},$$

then $Y \in \mathscr{C}_p$ and

$$\|d_{T^*,T}(Y)+\hat{S}\|_p = \left\| \begin{bmatrix} 0 & d_{A^*,B^*}(X)+S \\ 0 & 0 \end{bmatrix} \right\|_p$$
$$= \|d_{A^*,B^*}(X)+S\|_p \geqslant \|S\|_p = \|\hat{S}\|_p.$$

Recall that $\mathscr{C}_p$, $1<p<\infty$, is a uniformly convex space, that every non-trivial $\hat{S}\in\mathscr{C}_p$ is a smooth point, and that the support functional $D_{\hat{S}}$ is given by

$$D_{\hat{S}}(Z) = \operatorname{tr}\left(\frac{\tilde{S}Z}{\|\tilde{S}\|_q}\right)$$

for all $Z\in\mathscr{C}_p$. Replacing Z by $d_{T^*,T}(Y)$, we have now from Lemma 1 that (2) is satisfied if and only if $D_{\hat{S}}(d_{T^*,T}(Y))=0$, or, if and only if $\operatorname{tr}(\tilde{S}d_{T^*,T}(Y))=0$. Choose Y to be the rank one operator $f\otimes g$ for some arbitrary elements f and g in $H\oplus H$. Then

$$\operatorname{tr}(\tilde{S}\delta_{T^*,T}(Y)) = \operatorname{tr}(\tilde{S}(T^*Y-YT)) = \operatorname{tr}((\tilde{S}T^*-T\tilde{S})Y)=0$$

(respectively,

$$\operatorname{tr}(\tilde{S}\triangle_{T^*,T}(Y)) = \operatorname{tr}(\tilde{S}(T^*YT-Y)) = \operatorname{tr}((T\tilde{S}T^*-\tilde{S})Y)=0)$$

implies that

$$(d_{T,T^*}(\tilde{S})f,g)=0 \iff \tilde{S}\in\ker d_{T,T^*}.$$

Suppose conversely that $\tilde{S}\in\ker d_{T,T^*}$. Since $\tilde{S}Y$ and $\tilde{S}d_{T^*,T}(Y)$ are trace class,

$$\operatorname{tr}(\tilde{S}\delta_{T^*,T}(Y)) = \operatorname{tr}(\tilde{S}T^*Y-\tilde{S}YT) = \operatorname{tr}(Y\tilde{S}T^*-YT\tilde{S}) = \operatorname{tr}(-Y\delta_{T,T^*}(\tilde{S}))=0$$

(respectively,

$$\operatorname{tr}(\tilde{S}\triangle_{T^*,T}(Y)) = \operatorname{tr}(\tilde{S}T^*YT-\tilde{S}Y) = \operatorname{tr}(YT\tilde{S}T^*-Y\tilde{S}) = \operatorname{tr}(Y\triangle_{T,T^*}(\tilde{S}))=0).$$

This, by Lemma 1, implies that inequality (2) holds. □

Lemma 3. *Let $A,B\in B(H)$ be such that* $\ker d_{A,B}\subseteq\ker d_{A^*,B^*}$. *Then $S\in\ker d_{A,B}$ if and only if $\tilde{S}\in\ker d_{T,T^*}$.*

Proof. $\Longrightarrow$. The hypothesis $\ker d_{A,B}\subseteq\ker d_{A^*,B^*}$ implies that if $d_{A,B}(S)=0$, then $d_{A^*,B^*}(S)=0$. We consider the case in which $d=\triangle$; the proof in the case in which $d=\delta$ is similar. If $ASB=S=A^*SB^*$, then $BS^*S=BS^*ASB=S^*SB$, i.e., $[B,|S|]=0$. Clearly, $(AUB-U)|\overline{\operatorname{ran}|S|}=0$. Since $B:\ker S\to\ker S$, we have also that $AUB=U$. Hence $|S|^{p-1}U^*=|S|^{p-1}B^*U^*A^*=B^*|S|^{p-1}U^*A^*$, i.e., $\tilde{S}\in\ker\triangle_{T,T^*}$.

$\Longleftarrow$. Conversely, if $\tilde{S}\in\ker d_{T,T^*}$, then $|S|^{p-1}U^*\in\ker d_{B^*,A^*}$, or, $U|S|^{p-1}\in\ker d_{A,B}\subseteq\ker d_{A^*,B^*}$. Once again we consider the case $d=\triangle$ (leaving the other case to the reader). The hypothesis $AU|S|^{p-1}B=U|S|^{p-1}=A^*U|S|^{p-1}B^*$ implies (upon arguing as above) that $B|S|^{2(p-1)}=|S|^{2(p-1)}B$ (i.e., $[B,|S|]=0$) and $AUB=U$. Hence $S\in\ker\triangle_{A,B}$. □

Lemma 4. *If* $A, B \in B(H)$ *are such that* $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$*, then*

$$\min\left\{\|d_{A,B}(X) + S\|, \|d_{A^*,B^*}(X) + S\|\right\} \geqslant \|S\|$$

for all $S \in \ker d_{A,B}$ *and* $X \in B(H)$.

Proof. If $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$, and if $S \in \ker d_{A,B}$, then (it is easily seen that) $\overline{\text{ran}\, S}$ reduces A, $\ker^{\perp} S$ reduces B, and $A_1 = A|\overline{\text{ran}\, S}$ and $B_1 = B|\ker^{\perp} S$ are normal operators. Letting $S_0 : \ker^{\perp} S \to \overline{\text{ran}\, S}$ be the quasi-affinity (i.e., S_0 is injective with dense range) defined by setting $S_0 x = Sx$ for each $x \in \ker^{\perp} S$, it then follows that $d_{A_1,B_1}(S_0) = 0 = d_{A_1^*,B_1^*}(S_0)$. Now let $A = A_1 \oplus A_2$ (w.r.t. the decomposition $H = \overline{\text{ran}\, S} \oplus (\overline{\text{ran}\, S})^{\perp}$), $B = B_1 \oplus B_2$ (w.r.t. the decomposition $H = \ker^{\perp} S \oplus \ker S$), and let $X : \ker^{\perp} S \oplus \ker S \to \overline{\text{ran}\, S} \oplus (\overline{\text{ran}\, S})^{\perp}$ have the matrix representation

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Then, since A_1 and B_1 are normal, and the norm of an operator matrix is greater than or equal to the maximum of the norm of the entries along the main diagonal of the matrix,

$$\|d_{A,B}(X) + S\| \geqslant \|d_{A_1,B_1}(X_{11}) + S_0\| \geqslant \|S_0\| = \|S\|$$

and

$$\|d_{A^*,B^*}(X) + S\| \geqslant \|d_{A_1^*,B_1^*}(X_{11}) + S_0\| \geqslant \|S_0\| = \|S\|.$$

This completes the proof of the lemma. □

Proof of the Theorem. Part (i) of the theorem is proved in Lemma 4, and part (ii) of the theorem follows from Lemmas 2 and 3. The 'if' part of (iii) of the theorem is obvious from Lemma 4; the 'only if' part is proved as follows. By Lemma 2, inequality (3) is satisfied if and only if $d_{B,A}(|S|^{p-1}U^*) = 0 = d_{B^*,A^*}(|S|^{p-1}U^*)$. Arguing as in the proof of Lemma 3 it is seen that $[B, |S|] = 0$ and $d_{A,B}(U) = 0$. Hence $d_{A,B}(S) = 0 = d_{A^*,B^*}(S)$. □

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